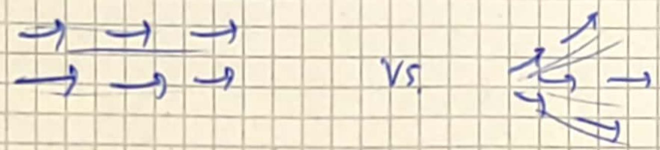
General Relativity $\boxed{\text{Week 4}}$

Definition: A connection on M is a map $\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ which is:

1) $C^\infty(M)$-linear in 1st argument: $\nabla_{X_1 + f \cdot X_2} Y = \nabla_{X_1} Y + f \cdot \nabla_{X_2} Y$
2) $\mathbb{R}$-linear in 2nd argument: $\nabla_X (Y_1 + a Y_2) = \nabla_X Y_1 + a \nabla_X Y_2$
3) Leibniz rule for 2nd argument: $\nabla_X (f \cdot Y) = X(f) \cdot Y + f \cdot \nabla_X Y$

Note: • $\nabla_X Y|_p$: "Directional" derivative of Y at p in the direction $X|_p$

• Lie derivative $L_X Y$ is not tensorial on X (depends on ∂X)



• If $Y \in \Gamma(M)$: ∇Y is a $(1,1)$-tensor field

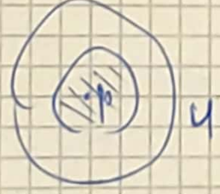
The connection is "local":

Lemma: Let $Y \in \Gamma(M)$, $Y|_U = 0$ on an open set U

Then $\forall X \in \Gamma(M)$: $\nabla_X Y|_U = 0$

Pf: $\forall p \in U$, let $f = 1$ in a neighborhood of p,

$f = 0$ on $M \setminus U$



Then $f \cdot Y = 0$ everywhere

$$\Rightarrow \underbrace{\nabla_X (f \cdot Y)|_p}_{=0} = \underbrace{X(f)|_p}_{0} \cdot Y|_p + \underbrace{f|_p}_{1} \cdot \nabla_X Y|_p \,. \quad \blacksquare$$

So: We can talk about ∇ "locally" around points

(i.e. only for vector fields defined on open sets, such as coordinate vector fields)

Christoffel symbols: In any local coordinate system,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \quad \text{or} \quad \Gamma^k_{ij} = dx^k\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)$$

Not a tensor field! - Depends on coordinates (But $\nabla - \bar{\nabla}$ is a tensor field)

~~Ex~~: ~~[illegible] connection~~

Note: $\boxed{(\nabla_X Y)^k = X^i \frac{\partial}{\partial x^i} Y^k + \Gamma^k_{ij} X^i Y^j}$

Ex: Flat connection on $\mathbb{R}^n$:

In Cartesian coordinates, $(\nabla_X Y)^k = X^i \frac{\partial}{\partial x^i} Y^k$

(so $\Gamma^k_{ij} = 0$ in Cart. coordinates, but not in other coordinate system e.g. polar coordinates)

Def: **Torsion**: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

(Note: $\nabla^*_X Y := \nabla_Y X + [X,Y]$: "Dual" connection)

- (1,2) - tensor field
- In coordinates $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$

∇ is torsion free iff $T \equiv 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji} \iff \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$

It suffices to hold in one coordinate system

We can extend ∇ to act on tensor fields by the requirement that

- $\nabla_X (f \otimes h) = (\nabla_X f) \otimes h + f \otimes (\nabla_X h)$

and
- $\mathrm{tr}(\nabla_X f) = \nabla_X (\mathrm{tr} f)$

For instance, from the above two properties:

For 1-forms: $\boxed{(\nabla_X \omega)_a = X^i \frac{\partial}{\partial x^i}\omega_a - \Gamma^k_{ia} X^i \omega_k}$

Levi-Civita connection:

Let (M,g) be a Lorentzian manifold. There exists a unique connection ∇ such that

1) ∇ is torsion free:
2) ∇ respects the metric:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in \Gamma(M)$$

$$\left(\Leftrightarrow \nabla_X g = 0 \quad \forall X \in \Gamma(M)\right)$$

Construction: In local coordinates:

$$X = \partial_i, \quad Y = \partial_j, \quad Z = \partial_k$$

$$\partial_i g_{jk} = g(\nabla_{\partial_i}\partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i}\partial_k)$$
$$= \Gamma^a_{ij} g_{ak} + \Gamma^a_{ik} g_{aj}$$

Permuting i,j,k and using $\Gamma^a_{ij} = \Gamma^a_{ji}$:

$$\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} = 2\Gamma^a_{ij} g_{ak} \Rightarrow \boxed{\Gamma^k_{ij} = \frac{1}{2} g^{ak}(\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij})}$$

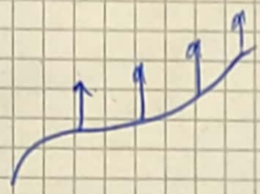
Rmks:
- Canonical connection
- Commutes with "function of g"
  e.g. the volume form,
  musical isomorphism!
  i.e. $(\nabla X)_\flat = \nabla(X_\flat)$

$\left(X_\flat = tr(g \otimes X) \right)$

So we will write $\nabla_a X_\beta$ without specifying if I apply the musical isomorphism before or after differentiation.

Parallel transport:

Let $\gamma : (a,b) \to M$ be a smooth curve.



I can consider vector fields along γ:

$$X \in \Gamma_\gamma : \quad t \longrightarrow X_t \in T_{\gamma(t)} M$$

Ex: $\dot\gamma$

Note: At points of self intersections of γ, $X \in \Gamma_\gamma$ cannot be extended to $\Gamma(M)$

But always: If I shrink the domain of γ around t so that no self intersection takes place, I can extend $X \in \Gamma_\gamma$ to $X \in \Gamma(M)$ (in a non-unique way).

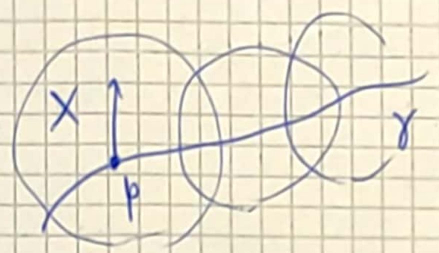
So I can talk about $\nabla_{\dot\gamma} X$

For any such extension: $\nabla_{\dot\gamma} X |_{\gamma(t)}$ is the same.

Proof: In local coordinates around $\gamma(t)$:

$$(\nabla_{\dot\gamma} Y)^k = \underbrace{\dot\gamma^i \partial_i Y^k}_{\frac{d}{dt} Y^k(\gamma(t))} + \Gamma^k_{ij} \dot\gamma^i Y^j$$

$$\text{So} \quad = \frac{d}{dt} Y^k + \Gamma^k_{ij} \dot\gamma^i Y^j \qquad \square$$



Def: $X \in \Gamma_\gamma$ is parallel transported if $\nabla_{\dot\gamma} X = 0$.

$\forall \xi \in T_pM$: $\exists!\ X \in \Gamma_\gamma$ s.t. $\begin{cases} \nabla_{\dot\gamma} X = 0 \\ X|_{t=0} = \xi \end{cases}$

$(p = \gamma(0))$

(So $X|_t = \mathbb{P}^{(\gamma)}_{0\to t}\, \xi$)

Pf: In local coordinates:

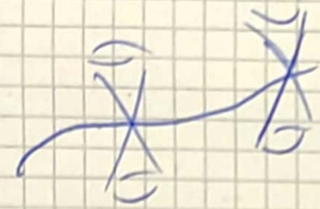
$$\begin{cases} \frac{d}{dt} X^k + \Gamma^k_{ij} \dot\gamma^i X^j = 0 \\ X^k|_{t=0} = \xi^k \end{cases}$$

Linear ODE, unique solution. (Covering the curve with a finite number of coordinate charts between $\gamma(0)$ and $\gamma(t)$ ...)

Parallel transport: $\mathbb{P}^{(\gamma)}: T_{\gamma(0)}M \longrightarrow T_{\gamma(t)}M$

<u>linear isometry</u>:

$$\nabla_{\dot\gamma} \langle X, Y \rangle = 0 \quad \text{if} \quad \nabla_{\dot\gamma} X = \nabla_{\dot\gamma} Y = 0$$

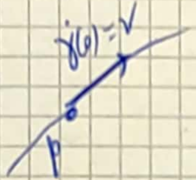


← Sends the light cone to the light cone!

<u>Geodesic</u>: $\nabla_{\dot\gamma} \dot\gamma = 0$

In Minkowski: Straight line.

<u>Thm</u>: $\forall p \in M$, $\forall v \in T_pM$: $\exists!$ <u>maximal</u> geodesic with $\gamma(0) = p$, $\dot\gamma(0) = v$



<u>Pf</u>: In local coordinates: $\ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0$

So: by existence and uniqueness of solutions for ODE ... ☺

<u>Rmk</u>: • If γ_1, γ_2 geodesics and $\left.\begin{array}{l} \gamma_1(T) = \gamma_2(T) \\ \dot\gamma_1(T) = \dot\gamma_2(T) \end{array}\right\}$ $\gamma_1 = \gamma_2$ on their common domain of definition

• $\nabla_{\dot\gamma} \dot\gamma = 0 \Rightarrow g(\dot\gamma, \dot\gamma) = \text{const}$

So it makes sense to talk about timelike / spacelike / null geodesics

Moreover: If X is a Killing vector field

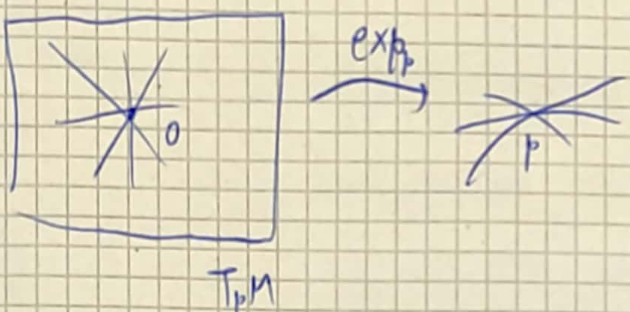
$$\Rightarrow \quad g(X, \dot{\gamma}) \text{ is } \underline{\text{constant}} \text{ along } \gamma$$

If I have enough symmetries: I can reduce the study of geodesics to a 1st order problem.

<u>Exponential map.</u>

Geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = v$

Let $\Omega_p := \{ v \in T_pM : \gamma_{p,v} \text{ is defined on } [0,1] \}$



Define $\exp_p : \Omega_p \longrightarrow M$

by $\exp_p(v) = \gamma_{p,v}(1)$

- Smooth map (since $\gamma_{p,v}$ depends smoothly on the initial data)
- $\exp_p(tv) = \gamma_{p,v}(t) = \gamma_{p,tv}(1)$

<u>Lem</u> $\quad d\exp_p \big|_{v=0} = id$

<u>Pf</u>: If $\sigma(t) = tv$ (straight line in T_pM)

$$\underbrace{\frac{d}{dt}\left(\exp_p(\sigma(t))\right)\Big|_{t=0}}_{\dot{\gamma}_{p,v}(t)\big|_{t=0} = v} = d\exp_p\big|_{\sigma(t)}(\dot{\sigma}(t))\Big|_{t=0} = d\exp_p\big|_{v=0}(v)$$

$\boxed{☺}$

So: By ~~[illegible]~~ inverse function theorem: $\exp_p$ is a local diffeo around $v = 0$.

| can use $\exp_p$ to define local coordinates around p

Normal coordinates around $p \in (M, g)$

Choose orthonormal frame $\{e_0, \dots, e_n\}$ in T_pM

So $\forall v \in T_pM$: $v = v^a e_a$

Then define coordinates $(x^0, \dots, x^n)$ around p so that

$$x^a(\exp_p(v)) = v^a \iff x^a(q) = (\exp_p^{-1}(q))^a$$

In those coordinates: $\gamma^k_{p,v}(t) = v^k t$ (geodesics through p are straight lines)

Proposition:

In normal coordinates around p: $(p = (0, \dots, 0))$

$$g_{\alpha\beta}(0) = \eta_{\alpha\beta}, \quad \partial_k g_{\alpha\beta}(0) = 0, \quad \Gamma^k_{\alpha\beta}(0) = 0$$

(But, in general: $\partial^e_{\alpha\beta} g_{\mu\nu} \neq 0$)